International Journal of Theoretical Physics, Vol. 22, No. 9, 1983

Causality between Preparation and Registration Processes in Relativistic Quantum Theory

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Received December 2, 1981

A causality postulate is considered which is based on the conception of systems that are prepared in some finite region of space-time and recorded in some other region. If these regions are spacelike separated, the recording apparatus should react as if no preparing apparatus were present, i.e., it should respond with at most some vacuum rate. The causality postulate is mathematically formulated within the framework of statistical theories. The connections with algebraic field theory are discussed and the relation between causality and spectral conditions is studied. General methods for constructing systems satisfying the causality postulate are given and applied in several examples.

1. INTRODUCTION

Einstein's principle of causality states that parts of an experiment performed in spacelike separated regions of space-time cannot influence each other. Therefore, to apply it to any particular physical theory one has to specify the way the parts of an experiment, their location in space-time, and the possible interactions between them are to be represented.

In this paper we shall consider a causality postulate which is based on the conception of systems that are prepared in some finite region of space-time and recorded in some other region. If these regions are spacelike separated, the recording apparatus should react as if no preparing apparatus were present, i.e., it should respond with at most some vacuum rate.

0020-7748/83/0900-0781$03.00/0

The preparing and recording devices are assumed to be given by their macroscopic descriptions as in a laboratory manual. "Microsystems" or structures characteristic of microscopic physics enter into the formulation of our postulate only as the carriers of a specific kind of interaction between macroscopic systems such that our postulate is a natural extension of the causality principles of macroscopic physics.

The Einstein principle of causality can also be applied to a pair of measuring devices. If these are spacelike separated and thus do not influence each other they may be combined into a correlation experiment which reproduces the statistics of the original devices as marginal distributions. Our formalization of this principle (called local coexistence) generalizes the local commutativity of algebraic quantum field theory (Haag and Kastler, 1964).

The conceptual independence of our coexistence and causality principles is clear from the observation that coexistence in contrast to causality becomes trivial in the classical case. The type of influence local coexistence or commutativity exclude is not based on the actual propagation of the systems the theory describes: A violation of one of these principles would not imply directly that it is possible to transmit signals faster than light.

In Section 2 we shall formulate our postulates in the framework of statistical theories in the sense of Ludwig (1977). In Section 3 we discuss in some detail the connections of this framework and our postulates with the algebraic approach to quantum field theory.

Section 4 deals with the relation between causality and spectral conditions. One consequence (nonexistence of counters) is analogous to a well-known corollary of the Reeh–Schlieder theorem (Reeh and Schlieder, 1961) but is derived here from rather different physical assumptions.

Section 5 summarizes basic constructions for systems satisfying Axiom C. It includes necessary and sufficient conditions for a net of local algebras to fit into this framework.

Section 6 gives some examples that are straightforward applications of the methods developed in Section 5.

2. AXIOMS

We begin with a short review of Ludwig's framework for statistical theories. It was devised originally as a starting point for axiomatic quantum mechanics (Ludwig, 1970, 1977, 1981).

Every individual experiment may be split into a preparation and a registration. (The splitting need not be unique.) Both parts are to be described in terms of macroscopic physics. For our purposes it is sufficient to consider only registration procedures with two possible outcomes denoted + and −. The basic quantities of the theory are the relative frequencies of the outcome "+" in a long series of experiments performed according to the same pair of procedures. Different preparing procedures may lead to the same frequencies for all registrations. An equivalence class under the resulting relation is called a (statistical) state. The corresponding equivalence classes of registration procedures are called effects. When a few more elementary assumptions concerning statistical mixtures of states and effects are made, the following representation theorem holds: There is a base normed Banach space B with base K (its dual is an order unit space B' with unit 1 and order interval $L=[0,1]$) such that the set of states may be identified with a subset $K_f \subset K$, the set of effects may be identified with a subset $L_f \subset L$ and the frequency function on $K_f \times L_f$ with the canonical bilinear form $\langle .,. \rangle$: $B \times B' \to \mathbb{R}$. Under this pairing K_f separates points of B' and L_f separates points of B.

A theory is called classical, if $B \cong L^1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ), and quantum mechanical if $B = \mathfrak{T}_h(\mathcal{H})$ (hermitian trace class operators) for some Hilbert space $\mathcal{H}$.

All subsequent axioms (including Axiom C) may now be formulated in terms of B, K_f, and L_f although they are usually justified in terms of preparing and measuring procedures (Gerstberger, 1980).

We assume that included in the instructions for every procedure is some information as to where, when, and in what orientation and state of motion the equipment is to be set up. Changing only this part of the instructions according to a Poincaré transformation defines a transformation in the set of procedures naturally preserving equivalence classes. We assume that its lift to K_f and L_f extends by continuity and linearity to transformations of B and B' (Axiom C1).

We assume that every laboratory procedure is to be performed in a finite space-time region. Finiteness in time means here that the apparatus, though it may have been built some time earlier, is active as a part of the experiment only for a finite time and then switched off. It is important to note that procedures in different, even disjoint, regions may still be equivalent statistically and thus belong to the same state. We define for every bounded open region $\sigma \subset \mathbb{R}^4 \mathcal{K}(\sigma) \subset K_f$ to be the set of states containing a preparing procedure that operates in σ. The corresponding set of effects will be denoted by $\mathcal{L}(\sigma)$. (Axioms C2, C3, C4).

There is one especially simple preparation procedure that consists of setting up no apparatus at all. An experiment with this preparation is performed by only triggering the measuring device at the prescribed time and reading the result some time later. The corresponding state will be called "vacuum" and denoted by W_0. Of course every experimental physicist knows that preparing "vacuum" is a good deal more complicated than described here and takes a lot of pumping and screening. However, this is also needed to perform any other state preparation. Otherwise one would not even get reproducible counting rates, making impossible the very definition of states. Thus the vacuum preparation should perhaps be described more precisely as "Do all the screening necessary, but do not set up any additional preparing apparatus." With this description of vacuum Axiom C5 is a direct application of Einstein's principle of causality.

The following notations for Minkowski space will be used in the sequel: $\mathcal{O}$ is the set of open sets in $\mathbb{R}^4$, $\mathcal{O}_f$ the set of bounded open sets. $\sigma \times \sigma'$ means that σ and σ' are (pointwise) spacelike. σ^c will denote the open spacelike complement: $\sigma^c = \cup\{\sigma' | \sigma' \in \mathcal{O}_f, \sigma' \times \sigma\}$. A net $\{g_i\}$ in the restricted Poincaré group $\mathcal{P}_+^\uparrow$ is said to go to spacelike infinity ($g_i \to s\infty$) if for $\sigma_1, \sigma_2 \in \mathcal{O}_f$ there is i_0 such that $g_i\sigma_1 \times \sigma_2$ for $i \geqslant i_0$. We then have the following axiom.

Axiom C. [Given: a base normed space (B, K) and its dual order unit space (B', L).]

(1) There is a $\sigma(B, B')$-continuous representation α of $\mathcal{P}_+^\uparrow$ by linear transformations of B such that $\forall_{g \in \mathcal{P}_+^\uparrow} \alpha_g(K) \subset K$. (The dual representation $\bar{\alpha}$ is defined by $\langle \alpha_{g^{-1}} W, F \rangle =: \langle W, \bar{\alpha}_g F \rangle$.)

(2) For every $\sigma \in \mathcal{O}_f$ there are sets $\mathcal{K}(\sigma) \subset K$, $\mathcal{L}(\sigma) \subset L$ such that

(a) $\mathcal{K}_f := \bigcup_{\sigma \in \mathcal{O}_f} \mathcal{K}(\sigma)$ separates points of B' and

(b) $\mathcal{L}_f := \bigcup_{\sigma \in \mathcal{O}_f} \mathcal{L}(\sigma)$ separates points of B.

(3) For $\sigma \in \mathcal{O}_f$ and $g \in \mathcal{P}_+^\uparrow$: (a) $\alpha_g \mathcal{K}(\sigma) \subset \mathcal{K}(g\sigma)$, (b) $\bar{\alpha}_g \mathcal{L}(\sigma) \subset \mathcal{L}(g\sigma)$.

(4) For $\sigma_1 \subset \sigma_2$: (a) $\mathcal{K}(\sigma_1) \subset \mathcal{K}(\sigma_2)$ and (b) $\mathcal{L}(\sigma_1) \subset \mathcal{L}(\sigma_2)$.

(5) There is a state $W_0 \in K$ (called vacuum) such that $\sigma_1 \times \sigma_2$, $W \in \mathcal{K}(\sigma_1)$, $F \in \mathcal{L}(\sigma_2)$ imply $\langle W, F \rangle = \langle W_0, F \rangle$.

A pair $(\mathcal{K}, \mathcal{L})$, $\mathcal{K}: \mathcal{O}_f \to \mathcal{P}(K)$, $\mathcal{L}: \mathcal{O}_f \to \mathcal{P}(L)$ satisfying these assumptions will be called a causal system.

The space $D := \overline{\mathrm{lin}\{1\} \cup \mathcal{L}_f}^{\|\cdot\|}$ is called the set of quasilocal observables in analogy with Haag and Kastler (1964). With every $\sigma \in \mathcal{O}$ we associate the weakly closed subspaces $\overline{\mathcal{K}}(\sigma) := [\{W_0\} \cup \bigcup\{\mathcal{K}(\tilde{\sigma}) | \sigma \supset \tilde{\sigma} \in \mathcal{O}_f\}]^{\perp\perp}$ and $\overline{\mathcal{L}}(\sigma) := [\{1\} \cup \bigcup\{\mathcal{L}(\tilde{\sigma}) | \sigma \supset \tilde{\sigma} \in \mathcal{O}_f\}]^{\perp\perp}$. The following proposition summarizes some elementary consequences of Axiom C:

Proposition 1. (1) W_0 is uniquely determined and Poincaré invariant.
(2) For $S \in B$, $T \in D$, and any sequence $g_n \in \mathcal{P}^{\uparrow}_{+}$ with $g_n \to s\infty$

$$\lim_{n\to\infty} \langle \alpha_{g_n} S, T\rangle = \langle S, 1\rangle\langle W_0, T\rangle$$

(3) Let g be a spacelike translation
(a) $S \in B, \alpha_g S = S \Rightarrow S = \langle S, 1\rangle \cdot W_0$
(b) $T \in D, \bar{\alpha}_g T = T \Rightarrow T = \langle W_0, T\rangle \cdot 1$

(4) (a) $\bigcap_{\sigma \in \mathcal{C}_f} \overline{\mathcal{K}}(\sigma^c) = \mathbb{R} W_0$ (there are no states at infinity)
(b) $\bigcap_{\sigma \in \mathcal{C}_f} \bar{\mathcal{L}}(\sigma^c) = \mathbb{R} \cdot 1$ (there are no effects at infinity).

Proof. (1) Let W_0, W_0' be vacua satisfying C5 and $g \in \mathcal{P}^{\uparrow}_{+}$. For any $F \in \mathcal{L}_f$, say $F \in \mathcal{L}(\sigma)$, pick $\sigma' \in \mathcal{C}_f$ with $\sigma' \times \sigma$ and $W \in \mathcal{K}(\sigma')$. By C5 and C3 with $g\sigma' \times g\sigma$ we have $\langle W, F\rangle = \langle W_0, F\rangle = \langle W_0', F\rangle = \langle \alpha_g W, \bar{\alpha}_g F\rangle = \langle W_0, \bar{\alpha}_g F\rangle = \langle \alpha_g W_0, F\rangle$. Since $\mathcal{L}_f$ separates points: $W_0 = W_0' = \alpha_g W_0$.

(2) This is trivial for lin $\mathcal{K}_f$ and lin $\mathcal{L}_f$ and extends to the norm-closures.

(3) Special case of (2) since $g^n \to s\infty$.

(4) For $F \in \mathcal{L}(\sigma)$ and $S \in \overline{\mathcal{K}}(\sigma^c)$: $\langle S, F\rangle = \langle S, 1\rangle \cdot \langle W_0, F\rangle$ by C5 and definition of $\mathcal{K}(\sigma)$. (4a) then follows from C2b. (4b) is analogous. ∎

Axiom C only formulates the consequences of Einstein's causality for statistical measurements. In an individual experiment it is impossible in general to decide whether the positive response of the registration apparatus was "caused" by a microsystem or by the random "response" of the apparatus to vacuum. This distinction can be made only if the vacuum rate of the apparatus is zero. Of course an experimental physicist always tries to make the vacuum rate of his recording instruments as small as possible. If he does not succeed in making it zero he will subtract from every counting rate the previously measured vacuum rate. The obtained result will serve him as a fair approximation of the rate he would have measured with the detector he did not quite manage to build. The problem with this procedure is of course that it may lead to negative "rates." Therefore the reinterpretation of rate differences in terms of a hypothetical apparatus is only possible for detectors represented by a particular kind of effects called "counters":

Definition. $F \in L$ is called a counter, if $\forall_{W \in K} \langle W, F\rangle \geq \langle W_0, F\rangle$. L_z will denote the set of counters.

For a counter F the hypothetical apparatus described above is represented by $F-\langle W_0, F\rangle 1 \in L$. All measuring devices commonly found in practice may be considered at least as close approximations of counters or correlation arrangements of counters and inverted counters $[F \leqslant \langle W_0, F\rangle 1]$. We shall say that a causal system $(\mathcal{K}, \mathcal{L})$ "contains many counters" if $\mathcal{L}_f \cap L_z$ still separates points of B. In contrast, it will be shown in Section 4 that no system satisfying the spectral condition contains any local counters at all.

We now formulate the postulate of local coexistence. In the framework of statistical theories the notion of coexistence of two effects describes the possibility of measuring the effects by the same apparatus: Suppose a measuring procedure with two binary displays is given which then leads to four different outcomes (denoted $++, +-, -+, --$). One may define corresponding effects F_{++}, F_{+-}, etc. by the procedure of applying the original apparatus but interpreting only the outcome "$++$" as a positive result. From this description it is clear that

$$\sum_{\varepsilon_1, \varepsilon_2 = \pm} F_{\varepsilon_1, \varepsilon_2} = 1$$

The effects $F_1 = F_{++} + F_{+-}$ and $F_2 = F_{++} + F_{-+}$ then describe measurements in which the second (or, respectively, the first) digit of the display is ignored. Two arbitrary effects F_1, F_2 are called coexistent if they admit a decomposition into effects $F_{\varepsilon_1, \varepsilon_2}$ satisfying the equations above. Coexistence is only a necessary but not a sufficient condition for the existence of a device measuring both effects. It is, however, the strongest condition that may be formulated in $\langle B, B'\rangle$, i.e., in terms of statistical equivalence classes of procedures. The relation of coexistence to the more widely known commutativity properties is described in Proposition 2 below.

The possibility of combining measuring devices in spacelike separated regions into a correlation experiment is now described by the following axiom.

Axiom CX (local coexistence). Let $\sigma_1 \times \sigma_2$, $F_i \in \mathcal{L}(\sigma_i)$, $i = 1,2$. Then there are effects $F_{\varepsilon_1, \varepsilon_2} \in \mathcal{L}(\sigma_1 \cup \sigma_2)$, $\varepsilon_1, \varepsilon_2 = \pm$, such that $F_1 = F_{++} + F_{+-}$, $F_2 = F_{++} + F_{-+}$, and $\Sigma F_{\varepsilon_1, \varepsilon_2} = 1$.

3. CONNECTIONS TO THE ALGEBRAIC APPROACH

The ideas expressed in Axioms C and CX bear a strong resemblance to the algebraic approach to quantum field theory. Therefore it may be useful to point out the relationship between the two approaches in some more detail.

The first difference to be mentioned concerns the style of axiomatic treatment. Ludwig's general approach to axiomatic formulations (which we follow) places a strong emphasis on directly observable elements of reality as a starting point for axiomatic constructions. While this motive is clearly present in the approach of Haag and Kastler, it seems to be partially superseded by the tendency to introduce mathematical structures (in particular algebraic structures, see below) which cannot be referred immediately to observations, but are rather justified by an appeal to traditions of theory construction and a desire to obtain strong mathematical consequences. Thus in the terminology of Ludwig the Haag–Kastler theory may be called an axiomatic theory which is not stated in the form of an "axiomatic basis." Of course this remark is not intended as a criticism of algebraic quantum field theory, since an appeal to accepted theories and the desire to obtain a rich mathematical theory are ingredients of any axiomatic approach. By attempting to construct an axiomatic basis for relativistic quantum theories we merely try to clarify some physical implications independent of algebraic assumptions, thus presenting a new point of view on some properties also known in the algebraic approach.

The problem of justifying the algebraic assumption also arises in axiomatic quantum mechanics. There, too, one may consider the postulate that the order unit space B' as constructed in Section 2 is isomorphic to the self-adjoint part of a C^*-algebra (or a Jordan algebra). But is is a hard and as yet unsolved problem to derive this postulate from assumptions on the linear and order structure of B', which describes the statistics of macroscopic measurements. [For attempts in this direction see Alfsen and Shultz (1978) and Araki (1980). In the axiomatic deduction of Hilbert space formalism (Ludwig, 1970, 1981) the status of algebraic assumptions is not apparent.] Sometimes a Jordan product is constructed via the assumption that a scale transformation on observables corresponds to a similar transformation on the elements of B'. However, this postulate (first expressed by von Neumann) would imply that the operator describing the mean values over the scale uniquely determines the yes–no effects corresponding to subsets of the scale (via the spectral resolution). But it can be demonstrated, e.g. Kraus (1974), that in typical measurement situations the yes–no effects of an observable are not even decision effects (i.e., projection operators in the Hilbert space model). Thus von Neumann's postulate is not an acceptable starting point for the construction of statistical theories in an axiomatic basis. More promising seems to be the idea of relating the multiplication operation to the possibility of executing "operations" one "after" the other. (An "operation" represents mathematically the measurement of an observable together with the resulting changes in the state of the systems). This approach seems to be favored by Haag and Kastler (1964) and Araki (1980), but the details have still to be worked out.

One characteristic feature of the axiomatic approach presented in this paper is that states and effects are treated much more symmetrically, i.e., the physically relevant states are also assumed to be given, just as the net of local observable algebras is assumed to be given in the algebraic approach. On the one hand this enables us to formulate as an axiom the impossibility of transmitting signals faster than light. On the other hand the problem of choosing the proper set of states (or representations) is excluded, or rather assumed to be solved, and only a condition on this choice is formulated. Some of the consequences of Axiom C that may be surprising at first glance (e.g., the uniqueness of the vacuum) arise in this way.

To make contact with the algebraic approach it is natural to identify the set of effects $\mathcal{L}(\sigma)$ associated with a region σ with the order unit interval of a local C^*-algebra $\mathcal{A}(\sigma)$, i.e., $\mathcal{L}(\sigma)=\{F\in\mathcal{A}(\sigma)|0\leqslant F\leqslant 1\}$. The inclusion relation of C4. is then to be interpreted as an inclusion of algebras and the space D coincides with the C^*-algebra of quasilocal observables.

The following theorem shows that the local commutativity (locality) of the algebraic theory becomes equivalent to our axiom of local coexistence, i.e., Axiom CX is a proper generalization of locality.

Proposition 2. Let $\mathcal{A}$ be a C^*-algebra, $L=[0,1]\subset\mathcal{A}$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be $*$-subalgebras of $\mathcal{A}$. Then $L\cap\mathcal{A}_i$ $(i=1,2)$ are pairwise coexistent if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ commute.

Proof. If $F_1, F_2\in L$ commute they are coexistent since the required decomposition may be constructed in the function calculus. ($F_{++}=F_1\cdot F_2$).

For the converse we may assume $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$ by choosing a faithful $*$-representation of $\mathcal{A}$.

Let $F_1\in L\cap\mathcal{A}_1$ and $F_2\in\overline{L\cap\mathcal{A}_2}^{w^*}$. Then F_1 and F_2 are coexistent: For any sequence $F^{(n)}\to F_2$ with $F^{(n)}\in L\cap\mathcal{A}_2$ we may choose a sequence of decompositions $F^{(n)}_{\varepsilon_1,\varepsilon_2}\in L$ with $\Sigma_{\varepsilon_1,\varepsilon_2=\pm}F^{(n)}_{\varepsilon_1,\varepsilon_2}=1$, $F_1=F^{(n)}_{++}+F^{(n)}_{+-}$; $F^{(n)}=F^{(n)}_{++}+F^{(n)}_{-+}$. By the weak *compactness of L we may find convergent subsequences yielding a decomposition of F_1, F_2. The same argument shows that $\overline{L\cap\mathcal{A}_1}^{w^*}$ and $\overline{L\cap\mathcal{A}_2}^{w^*}$ are coexistent. By Kaplansky's density theorem $\overline{L\cap A_i}^{w^*}=\overline{L\cap A_i}^{\|\|}$.

For projections coexistence implies commutativity. Hence $\mathcal{A}_1$ and $\mathcal{A}_2$, being generated by projections, commute. ■

The meaning of Axiom C within the algebraic approach is not so easily established, since there is no natural counterpart of the local states $\mathcal{K}(\sigma)$ in the axioms of that approach. The question should perhaps be posed in the following way: given a net of local algebras satisfying the Haag–Kastler axioms, is there a vacuum state W_0 and a family $\{\mathcal{K}(\sigma)\}_{\sigma\in\mathcal{C}_l}$ in the dual of

$\mathcal{Q} = \bigcup_{\sigma\in\mathcal{C}_f}\mathcal{Q}(\sigma)$ satisfying Axiom C together with $\{\mathcal{Q}(\sigma)\}$? Some sufficient conditions are relatively easy to give. For example if W_0 is faithful on every local algebra and satisfies certain cluster properties [to exclude observables at infinity (Bratteli and Robinson, 1979)] then a construction similar to Proposition 6 may be used to obtain sufficiently many local states. The space B of quasilocal states then coincides with the normal states in the vacuum representation. (Note that the local faithfulness of W_0 is equivalent to the nonexistence of local counters, which is related to the spectral condition by Proposition 4.) In this case the local states $\mathcal{K}(\sigma)$ are constructed by "indirect localization" in the sense of Section 5. In general, however, it is not clear whether the dual B' of the space $B=\overline{\text{lin}\bigcup_{\sigma\in\mathcal{C}_f}\mathcal{K}(\sigma)}^{\|\cdot\|_{D'}}$ of quasilocal states thus obtained from a given vacuum W_0 is again a C^*-algebra (hence a W^*-algebra). This would be desirable for the consistency of algebraic assumptions and amounts to the assertion that local operations in the sense of Haag and Kastler transform quasilocal into quasilocal states (although they do not, in general, transform local into local states).

Note that this assumption does not refer to the whole net of local states $\mathcal{K}(\sigma)$ but only to the closed subspace of D' generated by these states. Thus it may be formulated in the same setting as Axiom C. In order to express Axiom C in the algebraic approach we shall therefore consider the following postulate.

Postulate A. Let B and α be as in C1 and $\mathcal{L}(\sigma)\subset L$ for $\sigma\in\mathcal{C}_f$. These objects are said to satisfy Postulate A, if B is the self-adjoint part of the predual of a W^*-algebra $\mathfrak{M}$, $\mathcal{L}(\sigma)=\{F\in\mathcal{Q}(\sigma)|0\leqslant F\leqslant 1\}$ for some $*$-subalgebra $\mathcal{Q}(\sigma)\subset\mathfrak{M}$ and $\mathcal{L}(\sigma)$ satisfy all parts of Axiom C referring only to this family.

Assuming this postulate, the meaning of Axioms C and CX for the algebraic approach is completely clarified by Propositions 6 and 2.

It is clear that the stage of development of the theory presented in this paper leaves a lot to be desired compared to the level of sophistication that has been achieved in the algebraic approach. It seems to be very promising to study some of the physical ideas expressed in that approach in our setting, perhaps using analogous techniques. The importance of "physical equivalence" of causal systems may be mentioned. The concept of "operations" also admits a natural transcription into our approach. Preparations may be conceived as particular operations applied to the vacuum. Since operations carried out in spacelike separated regions should not influence each other, this leads to an axiom about the possibility of preparing "product" states by letting two preparing apparatuses operate in spacelike separated regions. This is a natural requirement for many-particle theories and may imply properties like the additivity of the spectrum of translations. Studying asymptotic approximations between a "free" and an "interacting" causal system results in a formulation of scattering theory in which geometric features play a prominent role. The authors hope to study at least some of these problems in subsequent papers.

4. CAUSAL SYSTEMS AND SPECTRAL PROPERTIES OF THE TRANSLATIONS

Throughout this chapter we assume a quantum mechanical description of the carriers of the interaction between the preparing and recording devices:

Axiom Q. There is a countable set of pairwise orthogonal projection operators $\{P_i\}_{i\in\mathbb{N}}$ with $\Sigma_i P_i = 1$ in a Hilbert space $\mathcal{H}$ and

$$K = \{W\in\mathcal{B}(\mathcal{H})\,|\,W\geqslant 0, \operatorname{tr} W = 1, \forall i[W, P_i] = 0\}$$

$$L = \{F\in\mathcal{B}(\mathcal{H})\,|\,0\leqslant F\leqslant 1, \forall i[F, P_i] = 0\}$$

The bilinear form is given by $\langle W, F\rangle = \operatorname{tr}(W\cdot F)$. The P_i are called superselection projectors.

As a consequence of Axiom C there is a unitary representation U (up to a multiplier) of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$ such that

$$\alpha_g(W) = U_g W U_g^*, \qquad \bar{\alpha}_g(F) = U_g F U_g^*, \quad \text{and} \quad \forall i\big[U_g, P_i\big] = 0$$

In strengthening a result of Proposition 1 the vacuum $W_0\in K$ turns out to be a pure state, i.e., there is a $\psi_0\in\mathcal{H}$ with $W_0 = P_{\psi 0}$. We have $\forall_{g\in\mathcal{P}_+^\uparrow} U_g\psi_0 = \psi_0$.

Proof. Since W_0 commutes with U_g for all $g\in\mathcal{P}_+^\uparrow$ and P_i for all $i\in\mathbb{N}$, this is also true for the spectral projections of W_0. With Proposition 1.3.a we have $W_0 = \lambda P_0$ for a finite-dimensional projection P_0. Since the restriction of the unitary representation of $\mathcal{P}_+^\uparrow$ to $P_0\mathcal{H}$ is trivial every one-dimensional projection $P_{\psi_0}\leqslant P_0$ commuting with all P_i gives $W_0 = P_{\psi_0}$. ■

It may be mentioned that all the following results of this chapter remain valid if, instead of assuming Q, only a representation of the Banach spaces B, B' by trace class and bounded Hilbert space operators, respectively, is given such that the representation of the subgroup of space-time translations of $\mathcal{P}_+^\uparrow$ is unitarily implemented.

The subgroup of space-time translations determines uniquely a projection valued measure E on $\mathbb{R}^4$ such that for all $x \in \mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$

$$U_x = \int e^{i\langle p, x\rangle} dE(p) \qquad \left(\langle p, x\rangle = \sum_\nu p^\nu x_\nu\right)$$

We discuss the connection of the energy-momentum observable given by E with the causal structure $(\mathcal{K}, \mathcal{L})$. From Proposition 1.3.b it already follows that effects of the form $F = \int f(p)\, dE(p)$ cannot represent real apparatuses such that $F \in \mathcal{L}(\sigma)$ with $\sigma \in \mathcal{O}_f$. The following proposition states a similar result.

Proposition 3. Assume Axioms C and Q. Let $x \in \mathbb{R}^4$ be a spacelike translation and A the unbounded self-adjoint operator $A = \int e^{|\langle p, x\rangle|} dE(p)$.
(1) If $W \in \mathcal{K}_f$ with $\operatorname{tr}(WA^2) < \infty$, then $W = W_0$.
(2) If $F \in \mathcal{L}_f$ with $\|AFA\| < \infty$, then $F = 0$.

Proof. $A^{-1}U_{tx} := \int \exp(-|\langle p, x\rangle| + it\langle p, x\rangle) dE(p)$ as a function of t is analytic in the strip $|\operatorname{Im} t| < 1$. Hence for $W \in \mathcal{K}_f$ satisfying the assumption of (1) and $F \in \mathcal{L}_f \operatorname{tr}(WU_{tx}^* F U_{tx}) = \operatorname{tr}(AWA \cdot A^{-1}U_{-tx} F A^{-1} U_{tx})$ is analytic for $|\operatorname{Im} t| < 1$ and equals $\operatorname{tr}(W_0 F)$ for sufficiently great t with $\operatorname{Im} t = 0$. Thus $\operatorname{tr}(WF) = \operatorname{tr}(W_0 F)$. The proof of (2) is quite similar. ■

(1) states that momentum distributions of states which can be prepared in finite space-time regions decrease slower than exponentially in spacelike directions. Likewise the sensitivity of recording devices in finite space-time regions decreases slower than exponentially for increasing momenta.

The support of the projection valued measure E is characterized by the decomposition of the representation of $\mathcal{P}_+^\uparrow$ into irreducible parts and the resulting mass distribution. As is already indicated by Proposition 3 there is a connection between the causal structure and the support of E. This is made more explicit in the following proposition.

Proposition 4. Assume Axioms C and Q. Let $\Gamma := \{p \in \mathbb{R}^4 \mid p_0 \geq |\mathbf{p}|\}$ denote the closed forward light cone and

$$H_\varepsilon = \{p \in \mathbb{R}^4 \mid \langle p, p\rangle = p_\varnothing^2 - \mathbf{p}^2 \leq -\varepsilon\} \qquad (\varepsilon > 0)$$

(1) $F \in \mathcal{L}_f$, $E(\Gamma \setminus \{0\})F = F$ imply $F = 0$.
(2) $F \in \mathcal{L}_f$, $E(H_\varepsilon)F = F$ imply $F = 0$.

Proof. In both cases we have $\langle W_0, F\rangle = \operatorname{tr}(W_0 E(\{0\})F) = 0$. If $W \in \mathcal{K}_f$, the function $\phi(x) := \operatorname{tr}(WU_x^* F U_x)$ thus has support in a closed double cone with compact base. If $W = \Sigma_n \lambda_n P_{\varphi n}$ is the spectral representation, the function

$$\phi_{n,\psi}(x):=\langle\psi|F^{1/2}U_x\varphi_n\rangle=\int e^{i\langle p,x\rangle}\langle F^{1/2}\psi|dE(p)\varphi_n\rangle$$

has support in the same double cone for all $n\in\mathbb{N}$, $\psi\in\mathcal{H}$. Moreover $\phi_{n,\psi}$ is the Fourier transform of a complex valued measure on $\mathbb{R}^4$, the support of which is contained in Γ or H_ε according to the assumptions of (1) or (2), respectively. By the following two lemmas either assumption implies $\phi_{n,\psi}\equiv 0$ hence $\phi(0)=\mathrm{tr}(WF)=0$ and $F=0$ since $\mathcal{H}_f$ separates points.

Lemma 1. If $T\in\mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution with support in a closed generating proper cone Γ, $\mathrm{supp}\hat{T}\neq\mathbb{R}^n$ implies $T=0$.

This lemma is a corollary of the "edge of the wedge" theorem (Streater and Wightman, 1964; theorem 2.17).

Lemma 2. Let $T\in\mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution, $\theta\in\mathbb{R}^n$. Suppose that supp T lies outside an open cylinder Z with axis $\theta(Z+\mathbb{R}\theta\subset Z)$ and $\mathrm{supp}\hat{T}\cap\{x\,|\,|\langle x,\theta\rangle|\leqslant\lambda\}$ is compact for all $\lambda\in\mathbb{R}$. Then $T=0$.

Proof. Let $f\in\mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\hat{f}$ compact. Let $S_f\in\mathcal{S}'(\mathbb{R}^n)$ be defined by $S_f(\varphi):=\int dt\,f(t)\varphi(\theta\cdot t)$. The Fourier transform of S_f is the function

$$\hat{S}_f(x)=\hat{f}(\langle x,\theta\rangle)$$

Hence $\hat{S}_f\cdot\hat{T}$ is well defined and has compact support. On the other hand $\mathrm{supp}(S_f*T)\subset\mathrm{supp}S_f+\mathrm{supp}T=\mathbb{R}\theta+\mathrm{supp}T$ does not intersect the cylinder Z. Since S_f*T as the Fourier transform of $\hat{S}_f\cdot\hat{T}$ is an analytic function it must be 0. By letting $\mathrm{supp}\hat{f}$ grow large one concludes $T=0$. ■

In quantum field theory one usually assumes the spectral condition.

Spectral condition: The support of the projection valued measure E is contained in the closed forward light cone.

Corollary. Assume C, Q and either the spectral condition or purely tachyonic spectrum of E (i.e., $\mathrm{supp}E\subset H_\varepsilon\cup\{0\}$, $\varepsilon>0$). Then

(1) $\mathcal{L}_f\cap L_z\subset\mathbb{R}\cdot 1$.

(2) If $\sigma\in\mathcal{O}_f$ and $F_\nu\in\mathcal{L}(\sigma)$, $\nu=1,2,\ldots,$ with $\langle W_0,F_\nu\rangle\to 0$ then $\langle W,F_\nu\rangle\to 0$ for all $W\in K$.

Proof. (1) If $F\in\mathcal{L}_f\cap L_z$ Proposition 4 may be applied to $F'=F-\langle W_0,F\rangle\cdot 1$.

(2) The proof consists in the observation that the set of localized effects $\mathfrak{L}(\sigma)$ can be assumed weakly closed and hence compact. ■

The spectral condition does not contradict the introduction of causal systems $(\mathfrak{K}, \mathfrak{L})$ as is shown by Example 6.3 (Section 6 below). However, in a causal theory with spectral condition all effects which can be recorded in finite regions of space-time are not counters but have nonvanishing vacuum probability rate which can be increased as well as decreased by appropriately prepared microsystems.

Results related to Proposition 4 and its corollary may be found in (Hegerfeldt, 1974; 1980). The nonexistence of counters is a well-known phenomenon in quantum field theory. It is a consequence of the Reeh–Schlieder theorem, which is proved rather similarly to Proposition 4. The physical presuppositions of that theorem are, however, quite different in that it makes use of ideas related to local algebras of fields rather than Axiom C.

It is an interesting question which spectra of E are compatible with Axiom C. For models containing counters the preceding corollary together with Examples 6.2 and 6.4 gives a fairly complete answer. Example 6.2 demonstrates in particular that the assumption $\varepsilon > 0$ for H_ε in Proposition 4 is essential.

For classical theories the exclusion of spacelike momenta is a direct consequence of causality. By 6.2. this is no longer true for quantum mechanics. Here the exclusion of tachyonic representations is motivated only by appeal to quantum-classical analogy. Whether it is possible to exclude "proper" tachyons ($\varepsilon > 0$) by Axiom C alone is as yet unknown.

Examples 6.3 and 6.5 show the compatibility of Axiom C with the spectral condition. In particular it is possible to construct causal systems for the irreducible representations of $\mathfrak{P}_+^\uparrow$ for $m > 0$ and the irreducible representations of the full Poincaré group $\mathfrak{P}$ for $m = 0$ and finite spin.

5. CONSTRUCTIONS

In many cases the families $\mathfrak{K}$ and $\mathfrak{L}$ may be enlarged by idealized elements without violation of Axiom C. For any family $\mathfrak{K}$ or $\mathfrak{L}$ and $\sigma \in \mathcal{C}$ we define

$$\mathfrak{K}^c(\sigma) := \{F \in L \,|\, \forall_{\sigma' \times \sigma} \forall_{W \in \mathfrak{K}(\sigma')} \langle W, F\rangle = \langle W_0, F\rangle\}$$

$$\mathfrak{L}^c(\sigma) := \{W \in K \,|\, \forall_{\sigma' \times \sigma} \forall_{F \in \mathfrak{L}(\sigma')} \langle W, F\rangle = \langle W_0, F\rangle\}$$

[Equivalently: $\mathfrak{K}^c(\sigma) = L \cap (\mathbb{R}1 + \overline{\mathfrak{K}}(\sigma^c)^\perp)$ and $\mathfrak{L}^c(\sigma) = K \cap (\mathbb{R}W_0 + \mathfrak{L}(\sigma^c)^\perp)$.] $\mathfrak{K}^c(\sigma)$ is called the set of effects indirectly localized in σ by $\mathfrak{K}$.

The following statements are easily verified: If $(\mathcal{K}, \mathcal{L})$ form a causal system, then $\mathcal{K} \leq \mathcal{L}^c$, $\mathcal{L} \leq \mathcal{K}^c$ and $(\mathcal{K}, \mathcal{K}^c)$, $(\mathcal{L}^c, \mathcal{L})$, $(\mathcal{K}^{cc}, \mathcal{K}^c)$ etc. again satisfy Axiom C. (Here $\mathcal{K} \leq \mathcal{L}^c$ iff $\mathcal{K}(\sigma) \subset \mathcal{L}^c(\sigma)$ for all $\sigma \in \mathcal{O}_f$.) For any family $\mathcal{K}$ or $\mathcal{L}$: $\mathcal{K}^{ccc} = \mathcal{K}^c$, $\mathcal{L}^{ccc} = \mathcal{L}^c$. If causal systems on the same Banach space B are ordered by set inclusion in both families $\mathcal{K}$ and $\mathcal{L}$ the maximal systems are just those of the form $(\mathcal{K}^{cc}, \mathcal{K}^c)$. Since $\mathcal{K}^{cc}$ may differ from $\mathcal{L}^c$, a causal system is not in general contained in a unique maximal system. Maximal systems always have the diamond property:

$$\mathcal{K}(\sigma) = \mathcal{K}(\sigma^{\diamond}), \qquad \mathcal{L}(\sigma) = \mathcal{L}(\sigma^{\diamond}) \qquad (\text{with } \sigma^{\diamond} = \sigma^{cc})$$

Indirect localization suggests a construction of causal systems which is primarily based on only one of the families $\mathcal{K}$ or $\mathcal{L}$. Axiom C5 is then automatically satisfied for $\mathcal{K}^c$ (or $\mathcal{L}^c$). Since indirect localization is a naturally covariant operation only the separation property of $\mathcal{K}_f^c$ (or $\mathcal{L}_f^c$) has to be transcribed into a property of $\mathcal{K}$ (or $\mathcal{L}$).

The following two propositions are of this type.

Proposition 5. Let B and α_g be as in Axiom C1, $W_0 \in K$ invariant under α_g. Let $\mathcal{K}$ be a family satisfying Axioms C2a, C3a, C4a. Then there is a family $\mathcal{L}$ satisfying Axiom C together with $\mathcal{K}$ if and only if

$$\bigcap_{\sigma \in \mathcal{O}_f} \overline{\mathcal{K}}(\sigma^c) = \mathbb{R} W_0$$

Proof. The "only if" part is Proposition 1.4a. By the preceding remark it is sufficient to calculate

$$(\mathcal{K}_f^c)^{\perp} = \Bigg(\bigcup_{\sigma \in \mathcal{O}_f} \mathcal{K}^c(\sigma) \Bigg)^{\perp} = \bigcap_{\sigma \in \mathcal{O}_f} \{ L \cap (\mathbb{R} 1 + \overline{\mathcal{K}}(\sigma^c)^{\perp}) \}^{\perp}$$

$$\underset{(*)}{=} \bigcap_{\sigma \in \mathcal{O}_f} \{ \mathbb{R} 1 + \overline{\mathcal{K}}(\sigma^c)^{\perp} \}^{\perp} = 1^{\perp} \cap \bigcap_{\sigma \in \mathcal{O}_f} \overline{\mathcal{K}}(\sigma^c)^{\perp\perp}$$

$$= 1^{\perp} \cap \mathbb{R} W_0 = \{0\} \qquad \blacksquare$$

For the equality $(*)$ one has to use that in an order unit space every subspace containing 1 is already positively generated. The corresponding statement for the base normed space B and W_0 is false as is the dual of Proposition 5. A counterexample is given in 6.6.

Under the additional assumption of Postulate A the dual proposition is true (independently of local commutativity):

Proposition 6. Let $B, \alpha, \mathfrak{L}$ satisfy Postulate A. Let W_0 be an α-invariant state. Then a family $\mathfrak{K}$ satisfying Axiom C together with $\mathfrak{L}$ exists if and only if

$$\bigcap_{\sigma\in\mathcal{O}_f} \bar{\mathfrak{L}}(\sigma^c) = \mathbb{R}1$$

Proof. Again one direction is Proposition 1.4.b and it suffices to prove that $\mathfrak{L}_f^c$ separates points of B'.

Let π: $\mathfrak{M}\to\mathfrak{B}(\mathfrak{H})$ be a faithful W^*-representation in which the vacuum W_0 is represented by a density matrix $\overline{W}_0$. Then, for $\sigma\in\mathcal{O}_f$ and any unitary operator $U\in\pi(\bar{\mathfrak{L}}(\sigma^c))'$ the state W_U: $A\mapsto \mathrm{tr}(\overline{W}_0 U^*\pi(A)U)$ is in $\mathfrak{L}^c(\sigma)$. Thus it suffices to prove that the states $U\overline{W}_0U^*$ with $U\in\mathfrak{D} := \bigcup_{\sigma\in\mathcal{O}_f}\pi(\bar{\mathfrak{L}}(\sigma^c))'$ and U unitary separate points of $\mathfrak{B}(\mathfrak{H})$.

The set $\mathfrak{D}$, as the union of an increasing net of $*$-algebras is a $*$-algebra. To show that it is weakly dense in $\mathfrak{B}(\mathfrak{H})$ it suffices to calculate $\mathfrak{D}'$. Using that for $\sigma\in\mathcal{O}$ $\bar{\mathfrak{L}}_{\mathbb{C}}(\sigma) := \bar{\mathfrak{L}}(\sigma)+i\bar{\mathfrak{L}}(\sigma)$ is a W^*-subalgebra of $\mathfrak{M}$ by C4b and A, we have

$$\mathfrak{D}' = \bigcap_{\sigma\in\mathcal{O}_f}\pi(\bar{\mathfrak{L}}(\sigma^c))'' = \bigcap_{\sigma\in\mathcal{O}_f}\pi\big(\bar{\mathfrak{L}}_{\mathbb{C}}(\sigma^c)\big) = \pi\bigcap_{\sigma\in\mathcal{O}_f}\bar{\mathfrak{L}}_{\mathbb{C}}(\sigma^c) = \pi(\mathbb{C}\cdot 1_{\mathfrak{M}}) = \mathbb{C}1$$

Suppose now that $A\in\mathfrak{B}(\mathfrak{H})$ and $\mathrm{tr}(U\overline{W}_0U^*A)=0$ for all unitary operators in $\mathfrak{D}$. Then the same is true for all unitaries in $\mathfrak{D}''=\mathfrak{B}(\mathfrak{H})$: For $B_1\ldots B_n\in\mathfrak{D}$ self-adjoint: $U=e^{iB_1}\ldots e^{iB_n}\in\mathfrak{D}$. By taking derivatives:

$$\mathrm{tr}\big(\overline{W}_0[B_1,[B_2,\ldots[B_n,A]\ldots]]\big)=0$$

In this formula we may take weak limits on $B_1\ldots B_n$, i.e., the formula is valid for $B_1\ldots B_n\in\mathfrak{D}''$. Hence $\mathrm{tr}(\overline{W}_0e^{iB}Ae^{-iB})=0$ for all $B=B^*\in\mathfrak{B}(\mathfrak{H})$.

For $\dim\mathfrak{H}=\infty$, $\overline{W}_0$ has two different eigenvalues, say λ_1,λ_2 with eigenvectors φ_1,φ_2. Now let ψ_1,ψ_2 be arbitrary unit vectors. There exists two unitary operators U_1, U_2 with $U_1\varphi_i=\psi_i$, $U_2\varphi_1=\psi_2$, $U_2\varphi_2=\psi_1$, and $U_1\varphi = U_2\varphi$ for $\varphi\perp\varphi_1,\varphi_2$. By subtracting the equations $\mathrm{tr}(U_i\overline{W}_0U_i^*A)=0$ one obtains $(\lambda_1-\lambda_2)(\langle\psi_1,A\psi_1\rangle-\langle\psi_2,A\psi_2\rangle)=0$. Therefore A is a multiple of unity, which has to be zero. ■

A typical property of causal systems with many counters is that $\mathfrak{K}(\sigma)$ may be chosen to be an exposed face of K. The assumption that B be the predual of an algebra is introduced into the following proposition mainly to have a convenient description of faces of K.

Proposition 7. Let $(B, \alpha, \mathcal{K}, \mathfrak{L})$ satisfy Axiom C. Suppose that B is the predual of a W^*-algebra and that $\mathfrak{L}$ contains many counters: $(\mathfrak{L}_f \cap L_z)^\perp = \{0\}$.

Then there is a family $\{Q(\sigma)\}_{\sigma \in \mathcal{O}}$ of projections in L such that

(1) $\alpha_g Q(\sigma) = Q(g\sigma)$, $\langle W_0, Q(\sigma)\rangle = 0$, $Q(\sigma) = \vee\{Q(\tilde{\sigma}) | \sigma^\diamond \supset \tilde{\sigma} \in \mathcal{O}_f\}$;

(2) $\bigwedge_{\sigma \in \mathcal{O}_f} Q(\sigma^c) = 0$;

(3) $1 - Q(\mathbb{R}^4) =: Q_0$ is a minimal projection with $\langle W_0, Q_0\rangle = 1$;

(4) $\forall_{W \in \mathcal{K}(\sigma)} \langle W, Q(\sigma^c)\rangle = 0$.

Conversely, if a family $\{Q(\sigma)\}$ satisfies 1...3, a causal system with many counters is given by

$$\mathcal{K}^Q(\sigma) := \{W \in K | \langle W, Q(\sigma^c)\rangle = 0\}$$

$$\mathfrak{L}^Q(\sigma) := \{F \in L | [1 - Q(\sigma)] F [1 - Q(\sigma)] = [1 - Q(\sigma)]\langle W_0, F\rangle\}$$

Proof. Let $\mathfrak{L}_z(\sigma) = \{F - \langle W_0, F\rangle \cdot 1 | F \in \mathfrak{L}(\sigma) \cap L_z\}$, $Q(\sigma) = \operatorname{supp} \cup_{\tilde{\sigma} \subset \sigma} \mathfrak{L}_z(\tilde{\sigma})$. Then $\mathcal{K}^Q = \mathfrak{L}_z^c \geqslant \mathcal{K}$, $\mathfrak{L}^Q = \mathfrak{L}_z^{cc} \geqslant \mathfrak{L}_z$ and properties 2 and 3 are straightforward reformulations of the separation properties of $\mathcal{K}_f^Q$ and $\mathfrak{L}_f^Q$. ■

Construction and properties of $Q(\sigma)$ suggest an interpretation in terms of "propositions" or "questions" (Jauch, 1968). $Q(\sigma)$ may then be paraphrased as "(Some part of) the system passes through σ." Condition 2 may be read as "No system is infinitely extended" (i.e., no system always affects the spacelike complement of any finite region); and condition 3 means "A system affecting no region in space-time is the vacuum." If in addition $Q(\sigma) \wedge Q(\sigma^c) = 0$ one might speak of a true point-particle. In this case conditions 2 and 3 are equivalent. Proposition 5 shows that the spectral condition and conditions 1 and 2 are already inconsistent.

Examples of causal systems with many counters are given in 6.1, 6.2, and 6.4.

In the quantum mechanical case Proposition 7 characterizes causal systems for which $\mathcal{K}(\sigma)$ is generated by the pure states P_φ for φ in some linear submanifold of $\mathcal{H}$. If "linear" is weakened here to "real-linear" it becomes possible to describe causal systems satisfying the spectral condition. Real subspaces have proved to be a convenient tool in the construction of free fields (Araki, 1964). The same properties used there also allow the construction of causal systems. For any subset $M \subset \mathcal{H}$ we define states and effects over M as

$$K(M) = \operatorname{conv}\{|\varphi\rangle\langle\varphi| \,|\, \|\varphi\| = 1, \varphi \in M\}$$

$$L(M) = \operatorname{conv} L \cap \{\lambda 1 + i\mu(|\varphi\rangle\langle\psi| - |\psi\rangle\langle\varphi|) | \lambda, \mu \in \mathbb{R}; \varphi, \psi \in M\}$$

$M' := \{\psi \in \mathcal{H} \mid \forall_{\varphi \in M} \operatorname{Im}\langle\psi|\varphi\rangle = 0\}$ is called the symplectic complement of M.

The following lemma summarizes the properties of these objects needed in the sequel.

Lemma. Let $M, N \subset \mathcal{H}$; $M \subset N'$.

(1) $\forall_{W_1, W_2 \in K(M)} \forall_{F \in L(N)} \langle W_1, F\rangle = \langle W_2, F\rangle$

(2) $F_1 \in L(M), F_2 \in L(N) \Rightarrow F_1$ and F_2 are coexistent.

(3) If M is a real linear subspace, the following are equivalent:

(a) $K(M)$ separates points of $\mathcal{B}(\mathcal{H})$.

(b) $L(M)$ separates points of $\mathcal{T}(\mathcal{H})$.

(c) $\dim_{\mathbf{R}} M' \leqslant 1$.

Proof. (1) is trivial.

(2) It suffices to consider the extreme points of $L(M)$ [respectively, $L(N)$]. These may be written as $F = \lambda 1 + |\psi^+\rangle\langle\psi^+| - |\psi^-\rangle\langle\psi^-|$ with $\lambda = \|\psi^-\|^2, 1-\lambda = \|\psi^+\|^2, \langle\psi^+, \psi^-\rangle = 0$, and $\psi^+ + \psi^-, i(\psi^+ - \psi^-) \in \operatorname{lin}_{\mathbf{R}} M$ (respectively, $\in \operatorname{lin}_{\mathbf{R}} N$). If $\psi_1^{\pm}, \psi_2^{\pm}$ are the corresponding eigenvectors of F_1 and F_2 the condition $M \subset N'$ may be expressed by

$$\langle\psi_1^{\pm}, \psi_2^{+}\rangle = \overline{\langle\psi_1^{\mp}, \psi_2^{-}\rangle}$$

We now have to construct a $G = F_{++} \in \mathcal{B}(\mathcal{H})$ such that $0 \leqslant G \leqslant F_1$; $G \leqslant F_2$; $F_1 + F_2 \leqslant 1 + G$. Setting $G = F_1F_2 = F_2F_1$ on $\{\psi_1^{\pm}, \psi_2^{\pm}\}^{\perp}$ we may assume $\mathcal{H} = \operatorname{lin}_{\mathbf{C}}\{\psi_1^{\pm}, \psi_2^{\pm}\}$. Using a continuity argument if necessary $\psi_1^{\pm}, \psi_2^{\pm}$ may be assumed to be linearly independent.

If $0 \leqslant G \leqslant F_1$ and $F_1\psi_1^- = 0$: $G\psi_1^- = 0$ and by analogous arguments: $G\psi_2^- = 0, G\psi_1^+ = F_2\psi_1^+, G\psi_2^+ = F_1\psi_2^+$. These equations uniquely determine an operator G. It suffices to prove that $G \geqslant 0$ since the other inequalities follow by symmetry ($F_k \mapsto 1 - F_k$) and uniqueness of G. Self-adjointness of G follows from the relations for $\langle\psi_1^{\pm}, \psi_2^{\pm}\rangle$. Positivity of the eigenvalues may be seen from the characteristic polynomial of G calculated in the basis $\{\psi_1^{\pm}, \psi_2^{\pm}\}$.

(3) a $\Rightarrow$ c: For $\psi_1, \psi_2 \in M'$: $|\psi_1\rangle\langle\psi_2| - |\psi_2\rangle\langle\psi_1| = 0$. Hence $\psi_1 = 0$ or $\psi_2 = \lambda\psi_1$ with $\lambda \in \mathbb{R}$.

b $\Rightarrow$ c: By (1), $K(M')$ consists of at most one element, i.e., $\dim_{\mathbf{C}} M' \leqslant 1$. If $\varphi \in M'$ and $i\varphi \in M'$: $\varphi \in M^{\perp}$ such that $|\varphi\rangle\langle\psi|$ with $\psi \perp \varphi$ is not distinguished from 0 by $L(M)$. Hence $\varphi = 0$.

c $\Rightarrow$ a: Since $K(\overline{M}) \subset \overline{K(M)}^{\sigma(B, B')}$ we may assume $M = M''$ to be a real closed subspace. If $\dim M' = 0$, $M'' = \mathcal{H}$. If $M' = \mathbb{R}\psi_0$, $K(M)$ still contains all one-dimensional projections since $K(M) = K(e^{i\alpha}M) = K(\mathcal{H})\forall_{\alpha \in \mathbf{R}}$.

c $\Rightarrow$ b: analogous. ■

The proof of the following proposition is then straightforward.

Proposition 8. Let $\mathcal{H}$ be a Hilbert space, U: $\mathcal{P}_+^\uparrow \to \mathcal{U}(\mathcal{H})$ a unitary representation with invariant vector ψ_0. Let $\{M(\sigma)\}_{\sigma \in \mathcal{O}}$ be a family of real linear subspaces of $\mathcal{H}$ such that

(1) $U_g M(\sigma) = M(g\sigma)$; $\psi_0 \in M(\sigma)$; $M(\sigma) = \cup \{M(\tilde{\sigma}) | \sigma \supset \tilde{\sigma} \in \mathcal{O}_f\}$

(2) $\bigcap_{\sigma \in \mathcal{O}_f} M(\sigma)' = \bigcap_{\sigma \in \mathcal{O}_f} M(\sigma^c)'' = \mathbb{R}\psi_0$

Then $\mathcal{K}(\sigma) := K(M(\sigma))$ and $\mathcal{L}(\sigma) := L(M(\sigma^c)')$ satisfy Axiom C. If in addition $M(\sigma^c) \subset M(\sigma)'$ or $M(\sigma^c) \subset iM(\sigma)'$ the smaller family $\tilde{\mathcal{L}}(\sigma) := L(M(\sigma))$ still satisfies Axiom C. Moreover $\sigma_1 \times \sigma_2$, $F_i \in \tilde{\mathcal{L}}(\sigma_i)$ imply that F_1, F_2 are coexistent. In this case one of the conditons (2) is redundant.

In typical cases (6.5) $\tilde{\mathcal{L}}$ fails to satisfy CX and the weak closure of $\mathcal{L}(\sigma)$ contains neither projections nor counters.

6. EXAMPLES

6.1. The Classical Free Particle. Let Ω_0 be the set of timelike straight lines in Minkowski space. For $g \in \mathcal{P}_+^\uparrow$ let T_g be the transformation of Ω_0 induced by the action of g on Minkowski space. From its identification with a homogeneous space of $\mathcal{P}_+^\uparrow$, Ω_0 inherits a natural Boolean σ-algebra Σ of subsets and a T_g-invariant measure μ. Let Ω be the measure space obtained from Ω_0 by adjoining a T_g-invariant point ω_0 with $\mu(\{\omega_0\}) = 1$. For $\sigma \in \mathcal{O}$ let $\Omega(\sigma)$ be the set of lines intersecting σ. Thus $\Omega(\sigma) \cap \Omega(\sigma') = \varnothing$.

By Proposition 7 the following objects constitute a causal system: $B := L^1(\Omega, \Sigma, \mu)$, $(\alpha_g \rho)(\omega) := \rho(T_{g^{-1}}\omega)$, $W_0(\omega) = \delta_{\omega, \omega_0}$, $Q(\sigma) =$ characteristic function of $\Omega(\sigma)$.

Explicitly:

$$\mathcal{K}(\sigma) = \left\{ \rho \in L^1 | \rho \geq 0, \int \rho \, d\mu = 1, \operatorname{supp} \rho \subset \Omega(\sigma) \cup \{\omega_0\} \right\}$$

$$\mathcal{L}(\sigma) = \{ f \in L^\infty | 0 \leq f \leq 1, f(\omega) = f(\omega_0) \text{ a.e. for } \omega \notin \Omega(\sigma) \}$$

The same pattern of construction via $(\Omega, T_g, \Omega(\sigma))$ may be used to define a large class of classical causal systems, including, e.g., free electrodynamic fields that may be generated by current distributions in a finite space-time region.

Clearly, all of these models satisfy Axioms A and CX and contain many counters.

6.2. A Quantum Mechanical Model Containing Many Counters and Tachyons. Using the same notations as in 6.1, let $\mathcal{H} = \mathcal{L}^2(\Omega, \Sigma, \mu)$,

$(U_g\psi)(\omega)=\psi(T_{g^{-1}}\omega)$, and $Q(\sigma)$ the multiplication operator by the characteristic function of $\Omega(\sigma)$.

Again Proposition 7 defines a causal system. The spectrum of translations in this "Koopmanized" version of 6.1 may be calculated explicitly (Doplicher et al., 1968). Apart from a nondegenerate eigenvalue at 0, it is the complement of the open forward and backward light cones, absolutely continuous with respect to Lebesgue measure and of infinite multiplicity.

6.3. Algebras of Observables Generated by Local Algebras of Fields. Proposition 6 provides a criterion for the existence of causal systems for local algebras of observables. If the algebra of observables is generated by local algebras of fields $\mathcal{F}(\sigma)$ the condition of Proposition 6 may be implied by conditions on $\mathcal{F}(\sigma)$. In (Doplicher et al., 1969) the connection between local algebras of observables and local algebras of fields is studied and the following result is observed, which appears here as a corollary of Proposition 6.

Corollary. In addition to the assumptions of Proposition 6 let π: $\mathfrak{M}\to\mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathfrak{M}$ such that

$$\pi(\mathcal{L}(\sigma^c))\subset\mathcal{F}(\sigma)'\subset B(\mathcal{H})\quad\text{and}\quad\Big(\bigcup_{\sigma\in\mathcal{C}_f}\mathcal{F}(\sigma)\Big)'=\mathbf{C}1$$

Then a family $\mathcal{K}$ satisfying Axiom C together with $\mathcal{L}$ exists.

The assumptions of this corollary are fulfilled in the cases of free Bose and Fermi fields.

6.4. The following model is based on an irreducible representation of the full Poincaré group in which time reflection is also represented by a unitary operator. It contains many counters.

Let D be an irreducible representation of $SL(2,\mathbf{C})$ in H_s of class $(s,0)$ (s integer or half-integer), such that in $\mathcal{S}=\mathcal{S}(\mathbb{R}^4)\otimes H_s$ a natural representation of $\mathcal{P}_+^{\uparrow}$ is given. For $f, g\in\mathcal{S}$ let

$$\langle f,g\rangle=\langle f,g\rangle_+ +\langle f,g\rangle_-$$

with
$$\langle f,g\rangle_\pm=\int\Big\langle\hat f(p),D\Big(\frac{\tilde p}{m}\Big)\hat g(p)\Big\rangle_s d\mu_\pm,$$

$d\mu_\pm=\eta(\pm p_0)\delta(p^2-m^2)\,d^4p$ and $\tilde p=p_0\sigma_0+\mathbf{p}\cdot\boldsymbol{\sigma}$.

By this nonseparating inner product a Hilbert space H_1 with the canonical mapping α: $\mathcal{S}\to H_1$ may be constructed. The representation U of $\mathcal{P}_+^{\uparrow}$ induced by α on H_1 is unitary. We have $H_1=H_{1+}\oplus H_{1-}$ such that

$\langle f,g\rangle=\langle f,g\rangle_{\pm}$ on $H_{1\pm}$. Let $P_{\pm}$ be the corresponding projections. The representation of $\mathcal{P}_+^{\uparrow}$ is irreducible in $H_{1\pm}$. By $(\widehat{\Gamma f})(p)=D(f/m\sigma_2)\hat{f}(-p)$ for $f\in\mathcal{S}$, $f=p_0\sigma_0-\mathbf{p}\cdot\boldsymbol{\sigma}$ an antilinear involution Γ can be defined on H_1 with the properties

(i) $U_g\Gamma=\Gamma U_g$ for all $g\in\mathcal{P}_+^{\uparrow}$

(ii) $P_+\Gamma=\Gamma P_-$;

(iii) $\langle\Gamma f,\Gamma g\rangle=\overline{\langle f,g\rangle}$

(iv) $\langle f,(P_+-D(-1)P_-)\Gamma f'\rangle=0$ if $f\in\mathcal{D}(\sigma)$, $f'\in\mathcal{D}(\sigma')$ with $\sigma\times\sigma'$, $\mathcal{D}(\sigma)=\{f\in\mathcal{S},\ \mathrm{supp}\,f\subset\sigma\}$.

We now define $H=\mathbb{C}\psi_0\oplus H_1$ as the Hilbert space of the quantum system. Let $Q(\sigma)$ be the projection onto the closure of $\mathcal{D}(\sigma^{\diamond})$ in H. We have $U_gQ(\sigma)U_g^*=Q(g\sigma)$ and $\forall_{\sigma\in\mathcal{C}_l}\ Q(\sigma)=1-Q_0$. If $A=(P_+-D(-1)P_-)\Gamma$, property (iii) of Γ implies $AQ(\sigma^c)A^{-1}+Q(\sigma)\leqslant 1-Q_0$. Hence $\bigwedge_{\sigma\in\mathcal{C}_l}Q(\sigma^c)=0$ and $\{Q(\sigma)\}_{\sigma\in\mathcal{C}_l}$ defines a causal system via Proposition 7.

This construction works for any spin s. The treatment of the vacuum given in Proposition 7 may also be modified to make the projection Q_0 a superselection projector.

6.5. Example 6.4 can also be used to define a causal system for an irreducible representation of $\mathcal{P}_+^{\uparrow}$ with $m>0$, $p_0>0$ and spin s. (Thus the spectral condition will be satisfied.) Such an irreducible representation is given in the subspace H_{1+} of H_1. Let $H_+=\mathbb{C}\psi_0\oplus H_{1+}$ and define

$$M(\sigma)=\mathbb{R}\psi_0\oplus\{P_+f\,|\,f\in H_1\quad\text{with}\quad Q(\sigma)f=f\quad\text{and}\quad\Gamma f=f\}$$

By (i) $M(\sigma)$ is covariant under the representation of $\mathcal{P}_+^{\uparrow}$ in H_+. With (ii) it can be shown that $\overline{\bigcup_{\sigma\in\mathcal{C}_l}M(\sigma)\oplus i\mathbb{R}\psi_0}=H_+$. From (iii) and (iv) easily follows $M(\sigma^c)\subset M(\sigma)'$ for integer spin and $M(\sigma^c)\subset(iM(\sigma))'$ for half-integer spin. Hence by Proposition 8 $K(M(\sigma))$ and $L(M(\sigma))$ satisfy Axiom C.

For the construction of causal systems for a free photon we consider an irreducible representation of the full Poincaré group with mass $m=0$ in $\mathcal{H}=\mathbb{C}\psi_0\oplus[L^2(\mathbb{R}^4,\mu_0)\otimes\mathbb{C}^2]$, where μ_0 is the Poincaré-invariant measure on the forward light cone. Similarly to the case $m>0$ real linear subspaces $M(\sigma)$ with $M(\sigma^c)\subset M(\sigma)'$ can be defined such that the assumptions of Proposition 8 are fulfilled.

Taking direct integrals one may now construct causal systems for all representations with timelike spectrum.

6.6. The techniques of Proposition 8 also yield a counterexample to the dual of Proposition 5: Let the family $\{M(\sigma)\}_{\sigma\in\mathcal{C}}$ be as in 6.5. Set $\mathcal{L}(\sigma):=$

$L(M(\sigma))$. Then $\mathfrak{L}_f = \cup_{\sigma \in \mathcal{O}_f} L(M(\sigma))$ separates points of $\mathfrak{T}(\mathcal{H})$ since it even separates $\mathfrak{B}(\mathcal{H})$ by Proposition 8. Covariance is trivial. To see that $\mathfrak{L}$ contains no effects at infinity, let

$$\tilde{\mathfrak{K}}(\sigma) = \text{lin}\{i(|\varphi\rangle\langle\psi| - |\psi\rangle\langle\varphi|) | \varphi, \psi \in M(\sigma^c)'\}$$

Then $\tilde{\mathfrak{K}}(\sigma^c) \subset \mathfrak{L}(\sigma)^\perp$ and $\cup_{\sigma \in \mathcal{O}_f} \tilde{\mathfrak{K}}(\sigma)$ is dense in $\mathfrak{T}(\mathcal{H})$, hence

$$\bigcap_{\sigma \in \mathcal{O}_f} \bar{\mathfrak{L}}(\sigma^c) = \bigcap_{\sigma \in \mathcal{O}_f} (\mathbb{R} \cdot 1 + \mathfrak{L}(\sigma^c))^{\perp\perp} = \left(\bigcup_{\sigma \in \mathcal{O}_f} (1^\perp \cap \mathfrak{L}(\sigma^c)^\perp) \right)^\perp$$

$$= \left(\bigcup_{\sigma \in \mathcal{O}_f} (1^\perp \cap \tilde{\mathfrak{K}}(\sigma)) \right)^\perp = (1^\perp \cap \mathfrak{T}(\mathcal{H}))^\perp = \mathbb{R} 1$$

[$\tilde{\mathfrak{K}}(\sigma)$ plays the same role as $\overline{\mathfrak{K}}(\sigma)$ in Proposition 1, only that it is not in general positively generated.] The sets $\mathfrak{L}(\sigma)$ contain counters like, e.g., the projections P_φ onto $\varphi \in M(\sigma)$ with $\varphi \perp \psi_0$. Since U_g satisfies the spectral condition, this contradicts Proposition 4.

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